



THE CURVATURE FUNCTION METHOD FOR TWO-DIMENSIONAL SHAPE OPTIMIZATION UNDER STRESS CONSTRAINTS

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Abstract—The curvature function method for two-dimensional shape optimization under stress constraints is developed. This method uses curvatures along the boundary curve as the design variables. First it is shown that local curvature has a monotonic relation to stress. Based on this, a zero-order search direction can be defined to search for the optimum curvature function which achieves a fully stressed boundary. No sensitivity analysis is required, and the method is completely independent of the analysis techniques for calculating the stress. The resulting curve has C^2 continuity if the curvature function is continuous. Three design examples are presented.

INTRODUCTION

Most research in shape optimization is focused on interfacing structural analysis programs directly with first-order optimization algorithms (e.g. sequential linear programming [1–3] and CONMIN feasible direction algorithm). These algorithms require gradients of constraints on structural responses such as stress, displacement and natural frequencies. Therefore sensitivity analysis of the structural responses has been an important research topic in the field of shape optimization. Calculating sensitivity by finite difference is computationally expensive when the number of design variables is large. On the other hand, analytical methods for sensitivity calculation are computationally more efficient, but considerable coding is required by experts in the field to implement them, and the available design variables are still limited.

These difficulties in obtaining first-order information motivated the current work of developing the curvature function method presented here. There is a zero-order optimization method for two-dimensional shape optimization under stress constraints; only one structural analysis is needed for each iteration, no sensitivity analysis is required. Fully stressed design is used as the optimality criterion. Since it is shown that the local curvature has a monotonic relation to stress, a zero-order search direction can therefore be defined to search for the optimum curvature function which achieves a fully stressed boundary. Three design examples are presented. The first example is a cantilever beam problem which has an analytical solution for comparison. The other two examples

have appeared widely in the shape optimization literature. The second example is to find the profiles of a constant stress fillet of a tension bar. The last example is the design of a torque arm.

CURVATURE AND STRESS AT A TWO-DIMENSIONAL BOUNDARY

In a two-dimensional stress problem, the stress value at a boundary point is determined by two factors: (1) the nominal stress; and (2) the stress concentration effect. Nominal stress depends on the load and the amount of material carrying the load. Obviously, the nominal stress can be reduced by adding material and vice versa. Stress concentration depends on the smoothness of local geometry. Abrupt change in local geometry results in high stress concentration.

Boundary smoothness has been one of the key factors for the success of two-dimensional shape optimization problems under stress constraints. Researchers have used polynomial [2, 7, 8] or spline curves [3, 9–12], in which smoothness is a built-in feature, for boundary representation. However, there has not yet been an explicit measure of smoothness used in the optimization process.

Curvature of the boundary curve can be used for this purpose, since it is closely related to stress concentrated effects. Peterson [13] showed the well-known fact that the stress concentration factor at a notch or a shoulder fillet increases monotonically as the radius (inverse of curvature) of the notch or fillet decreases. The basis for Petersen's results were experiments. Recent research by Gao [14] gives

an analytical solution for stress concentration at slightly undulating surfaces, which shows the same monotonic relation.

Under the assumption that the stress at a boundary point depends only on the depth (the amount of material) under this point and the curvature at this point, it can be shown that varying a single boundary to minimize the area under it, subject to the maximum stress constraint, the stress constraint is active everywhere along the boundary [15]. The fully stressed design criterion is widely used as the optimality criterion in structural optimal design. In two-dimensional shape optimization, it is also intuitive that in order to minimize the amount of material, the cross-sections should be varied in an attempt to maintain a constant maximum stress at all cross-sections. The fully stressed design is used as the optimality criterion for the curvature function method being presented here.

The most intuitive way to achieve a fully stressed boundary is to add material if the stress is high and remove material if the stress is low. But simply adding material does not ensure a decrease in stress, since this may create sharp corners which cause stress concentration. On the other hand, the theorem below, which was proved in Ref. [15], states that when the positions of both end points of the boundary curve to be varied are fixed, decreasing curvature will also increase depth under the curve, which does ensure a decrease in stress.

Theorem. As shown in Fig. 1, let $y_1(x)$ and $y_2(x)$ be continuous for $x \in (x_0, x_f)$, with curvature functions $\kappa_1(x)$ and $\kappa_2(x)$, respectively. Let $y_1(x_0) = y_2(x_0)$, $y_1(x_f) = y_2(x_f)$. If $\kappa_1(x) \leq \kappa_2(x)$ for every $x \in (x_0, x_f)$, then $y_1(x) \geq y_2(x)$ for every $x \in (x_0, x_f)$.

With this strictly monotonic relation between curvature and stress, it is much more convenient to work in the curvature domain when trying to achieve the fully stressed boundary. The curvature is increased if the stress is lower than the target stress, and decreased if the stress is higher than the target stress.

The curvature function of classic analytic geometry in the x - y plane is [16]:

$$\kappa(x) = \frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}. \quad (1)$$

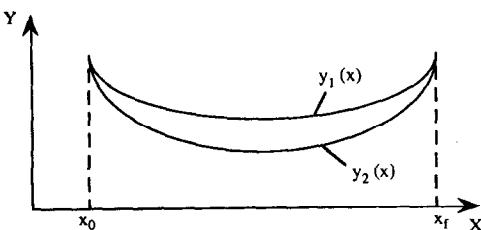


Fig. 1. If $\kappa_1(x) \leq \kappa_2(x)$, then $y_1(x) \geq y_2(x)$.

Equation (1) cannot be solved analytically. Simple numerical integration can be used to determine y given $\kappa(x)$ and two boundary conditions on the positions of both end points. Note that from eqn (1), the curve integrated from $\kappa(x)$ will have C^2 continuity if $\kappa(x)$ is continuous.

THE CURVATURE FUNCTION METHOD

The two-dimensional shape optimization problem is now transformed into one of finding the optimum curvature function $\kappa_c(x)$, which achieves the fully stressed boundary. The discrete version of eqn (1) is used to develop an algorithm applicable to digital computers. The boundary curve is discretized into n subsets, and its curvature function $\kappa(x)$ is represented by a corresponding piecewise linear function. Curvatures κ_i at the i th point, $i = 1, \dots, n-1$, are the control (decision) variables. Coordinates y_i at the i th point become the state (solution) variables. The task is to find a set of κ_i for which the resulting design satisfies the stress constraints with strict equality.

The procedure is iterative. A stress analysis is done on an initial design to check if the stress constraints are satisfied with strict equality. If not, a search direction is defined in the space of the κ_i and a new set of κ_i is generated. Then an analysis is performed on the new design to check if the stress constraints are met. This procedure continues in an iterative manner until the stress constraints are satisfied within a prescribed tolerance.

Zero-order search direction

From the discussion in the previous section, if the stress σ_i at a boundary point is too high (low), then the corresponding κ_i is reduced (increased) in the next iteration. A stress-ratio technique [17] can be applied to define $\Delta\kappa_i$ for the next iteration, assuming a linear relation between σ_i and κ_i :

$$\Delta\kappa_i = -\alpha p_i, \quad (2)$$

where

$$p_i \equiv \frac{(\sigma_i - \sigma_0)}{\max(\sigma_0, \sigma_i)}, \quad (3)$$

σ_0 is the maximum allowable stress and α is the step length. Note that $|p_i| \leq 1$, so α can also be interpreted as the maximum change in curvature between iterations.

In vector form, the curvature κ^k at the k th iteration is:

$$\kappa^k = \kappa^{k-1} - \alpha^{k-1} \mathbf{p}^{k-1}. \quad (4)$$

Note that the search direction \mathbf{p}^{k-1} is a zero-order search direction because only function values of the stress constraints are required, not derivatives. This

search direction is defined by engineering knowledge instead of purely numerical information.

Ideally the stress is constant everywhere along the boundary at the optimum. But practically speaking, the stress constraints near the two end points of the boundary might never be satisfied with strict equality since these two points are fixed, as required in the Theorem in the previous section. If the initial design represents an over-design, there might be small under-stressed zones at the end points; if the algorithm keeps iterating using eqn (4), the curvature at the understressed zones would approach infinity, which causes instability. On the other hand, if the problem is infeasible, there are always overstressed zones; if the algorithm keeps iterating using eqn (4), the curvature at the overstressed zones would approach minus infinity.

Therefore, a maximum curvature κ_{\max} is prescribed in the curvature function method to improve the stability. If the curvature is larger than κ_{\max} at a point on the boundary curve, then it is set to κ_{\max} and the point is an *acceptable understressed point*. If the curvature is less than $-\kappa_{\max}$, then the algorithm should stop and warn the user that the problem might be infeasible.

Performance index

In Fig. 2 S is the stress distribution curve of a design. A performance index $\Delta\sigma_{\text{abs}}$ is defined below to evaluate how close S is to the optimum stress distribution curve S_0 . Note that S_0 is a constant straight line at $\sigma = \sigma_0$.

$$\Delta\sigma_{\text{abs}} \equiv \frac{\sum_{i \in P} |\sigma_i - \sigma_0| \Delta x_i}{\sum_{i \in P} \Delta x_i},$$

where

$$P = \{1, 2, \dots, n\} \cap \{i | \kappa_i < \kappa_{\max}\}. \quad (5)$$

As shown in Fig. 2, geometrically $\Delta\sigma_{\text{abs}}$ is simply the sum of areas A , B and C (except the acceptable understressed points where $\kappa_i = \kappa_{\max}$), divided by the total length in x . It has the same units as stress and approaches zero as S approaches S_0 .

Adaptive step length procedure

To assure convergence of an algorithm, the step length α must produce a *sufficient decrease* in the

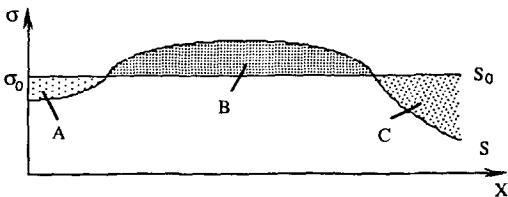


Fig. 2. Definition of the performance index $\Delta\sigma_{\text{abs}}$.

performance index [18]. A zero-order sufficient decrease criterion for the curvature function method is

$$\Delta\sigma_{\text{abs}}^k \leq \mu \Delta\sigma_{\text{abs}}^{k-1}, \quad \text{where } 0 < \mu < 1. \quad (6)$$

At the k th iteration, if eqn (6) is not satisfied, then step length α^k has to be adjusted. There can be two possible cases when eqn (6) is not satisfied: either α^{k-1} is too large and the k th iteration overshoots; or α^{k-1} is too small, making the convergence slow.

Consider the case when α^{k-1} is too small. In this case, $(\sigma_i^k - \sigma_0)$ and $(\sigma_i^{k-1} - \sigma_0)$ have the same sign since the k th iteration does not overshoot. Thus

$$\begin{aligned} & \sum_{i \in P} |\sigma_i^{k-1} - \sigma_0| - \sum_{i \in P} |\sigma_i^k - \sigma_0| \\ &= \sum_{i \in P} |(\sigma_i^{k-1} - \sigma_0) - (\sigma_i^k - \sigma_0)| \\ &\geq \left| \sum_{i \in P} (\sigma_i^{k-1} - \sigma_0) - (\sigma_i^k - \sigma_0) \right|, \end{aligned} \quad (7)$$

which implies

$$\Delta\sigma_{\text{abs}}^{k-1} - \Delta\sigma_{\text{abs}}^k \geq \frac{\left| \sum_{i \in P} (\sigma_i^{k-1} - \sigma_0) - (\sigma_i^k - \sigma_0) \right| \Delta x_i}{\sum_{i \in P} \Delta x_i}. \quad (8)$$

Define

$$\Delta\sigma_{\text{avg}} \equiv \frac{\sum_{i \in P} (\sigma_i - \sigma_0) \Delta x_i}{\sum_{i \in P} \Delta x_i}. \quad (9)$$

We have

$$\Delta\sigma_{\text{abs}}^{k-1} - \Delta\sigma_{\text{abs}}^k \geq |\Delta\sigma_{\text{avg}}^{k-1} - \Delta\sigma_{\text{avg}}^k|. \quad (10)$$

Similarly, in the case when α^{k-1} is too large, $\Delta\sigma_{\text{abs}}^{k-1} - \Delta\sigma_{\text{abs}}^k \geq |\Delta\sigma_{\text{avg}}^{k-1} + \Delta\sigma_{\text{avg}}^k|$. Therefore, when eqn (6) is not satisfied, if $\Delta\sigma_{\text{abs}}^{k-1} - \Delta\sigma_{\text{abs}}^k \geq |\Delta\sigma_{\text{avg}}^{k-1} - \Delta\sigma_{\text{avg}}^k|$, the step length is too small and should be increased in the next iteration; if $\Delta\sigma_{\text{abs}}^{k-1} - \Delta\sigma_{\text{abs}}^k \geq |\Delta\sigma_{\text{avg}}^{k-1} + \Delta\sigma_{\text{avg}}^k|$, the step length is too large and should be reduced in the next iteration.

Termination criterion

The user can specify a tolerance τ_σ for the amount of infeasibility of the stress constraints that will be tolerated, considering the precision of the stress data. The design is considered feasible if $\max(\sigma_i - \sigma_0) < \tau_\sigma$. The termination criterion is that the design is feasible and $\Delta\sigma_{\text{abs}} \leq \tau_\sigma$.

Finally the algorithm of the Curvature Function Method is presented below:

Algorithm: the curvature function method

input

Γ^0 initial boundary curve being optimized, defined in terms of x - y coordinates;
 σ_0 the maximum allowable stress;
 α^0 initial step length;
 κ_{\max} maximum curvature;
 τ_σ the infeasibility of the stress constraints that will be tolerated;

begin

constants $0 < \mu < 1$, $0 < \nu < 1$;
 iteration number $k \leftarrow 0$;
 calculate curvatures κ_i^0 of Γ^0 , $i = 1, 2, \dots, n$, from eqn (1);
 perform an analysis (e.g. finite element analysis) to evaluate σ_i^0 along Γ^0 ;
 calculate $\Delta\sigma_{\text{abs}}^0$ and $\Delta\sigma_{\text{avg}}^0$;

repeat

$$p_i^k \leftarrow \frac{(\sigma_i^k - \sigma_0)}{\max(\sigma_0, \sigma_i)}$$

$$\kappa_i^{k+1} \leftarrow \kappa_i^k - \alpha^k p_i^k;$$

if $\min(\kappa_i^{k+1}) < -\kappa_{\max}$ **then**

begin

print ("Warning: the problem might be infeasible!");
 stop;

end;

if $\kappa_i^{k+1} > \kappa_{\max}$ **then** $\kappa_i^{k+1} \leftarrow \kappa_{\max}$;

$k \leftarrow k + 1$;

integrate κ_i^k using eqn (1) with two end points fixed for Γ^k ;

perform an analysis to evaluate σ_i^k along Γ^k ;

calculate $\Delta\sigma_{\text{abs}}^k$ and $\Delta\sigma_{\text{avg}}^k$;

if $\Delta\sigma_{\text{abs}}^k \leq \mu \Delta\sigma_{\text{abs}}^{k-1}$ **then** $\sigma^k \leftarrow \alpha^{k-1}$;

else if $|\Delta\sigma_{\text{abs}}^{k-1} - \Delta\sigma_{\text{abs}}^k| \geq |\Delta\sigma_{\text{avg}}^{k-1} + \Delta\sigma_{\text{avg}}^k|$ **then**

$$\alpha^k \leftarrow \nu \alpha^{k-1};$$

$$\text{else } \alpha^k \leftarrow \frac{1}{\nu} \alpha^{k-1};$$

until $\max(\sigma_i^k - \sigma_0) < \tau_\sigma$ and $\Delta\sigma_{\text{abs}}^k < \tau_\sigma$

end.

Guidelines for choosing input parameters

Before applying this algorithm to design examples, guidelines for appropriate selection of input parameters Γ^0 , σ_0 , σ^0 , κ_{\max} and τ_σ to the curvature function algorithm are discussed. Γ^0 is the initial boundary curve being optimized, defined in terms of x - y coordinates. A good initial guess can accelerate the convergence of the algorithm. If the user has little

knowledge of what the optimum curve should be, a straight line between the two fixed end points should be a reasonable initial curve. σ_0 is the maximum allowable stress, which is provided by the requirement of the design.

α^0 is the initial step length. Though the step length will be adjusted by the adaptive step length procedure throughout the iterations, a good initial step length can also accelerate the convergence and reduce instability of the algorithm. As shown in eqns (2) and (3), the step length can also be interpreted as "the maximum change in curvature between iterations". So α^0 should be of the same order with the expected average curvature of Γ . Let x_0 and x_1 be the x coordinates of the two fixed end points of the boundary curve Γ . The radius of the circle which is tangent to both vertical lines $x = x_0$ and $x = x_1$ is $|x_1 - x_0|/2$. Thus the average curvature of Γ is estimated to be of the order $1/|x_1 - x_0|$. Therefore, a reasonable estimation of α^0 is

$$\alpha^0 = \frac{1}{|x_1 - x_0|} \quad (11)$$

Maximum curvature κ_{\max} also relates to the curvature of Γ . It was chosen to be $3\alpha^0$ for the design applications in this paper. τ_σ is the tolerance of the size of infeasibility of the stress constraints. It should not be smaller than the accuracy of the stress data. Finite element analysis was used to evaluate the stress data for the design applications in this paper, and τ_σ was set to be 5% of σ_0 . Constants μ and ν in the algorithm were chosen to be 0.8.

DESIGN APPLICATIONS

Three design applications using the curvature function method are presented in this section. The first example was to find the shape of a cantilever beam with constant maximum stress at all cross-sections. This problem has an analytical solution from beam theory for comparison. The second example, one of the most popular examples in the shape optimization literature, was to find the profiles of constant stress fillets of a tension bar for different stress concentration factors. The last example is the design application of a torque arm. This example has also appeared widely in the shape optimization literature. In these three examples, the curvature function method converged to within 5% tolerance after 8–15 iterations. Finite element analysis (using IDEAS 4.0 and ANSYS PC/linear 4.2) was used to evaluate the stress.

1. *Minimum area cantilever beam under bending*

The dimensions of a tapered cantilever beam are shown in Fig. 3. The boundary between points A and B was to be varied to minimize the area of the beam, under the constraint that the maximum stress cannot exceed σ_0 . In order to use a minimum amount of material, the cross-sections were varied in an attempt

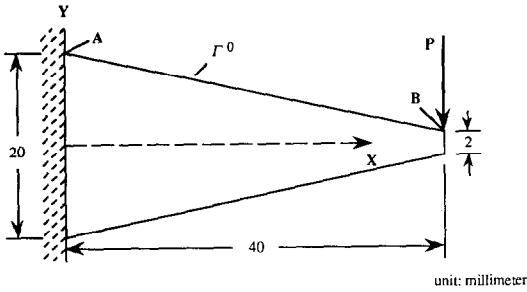


Fig. 3. A tapered cantilever beam.

to maintain a constant maximum stress at all cross-sections. From beam theory, the analytical solution for this problem is [19]:

$$Y = \sqrt{\frac{6P(L-x)}{b\sigma_0}}, \quad (12)$$

where in this example, load per unit width $P/b = 10 \text{ N mm}^{-1}$; $\sigma_0 = 10 \text{ MPa}$; length $L = 40 \text{ mm}$.

The curvature function method was used to solve this problem. The input parameters to the algorithm were:

$$\begin{aligned} \Gamma^0 &= \text{a straight line between point } A \text{ and } B; \\ \alpha^0 &= \frac{1}{40} \text{ mm}^{-1} \text{ [from eqn (11)];} \\ \kappa_{\max} &= 3\alpha^0 = \frac{3}{40} \text{ mm}^{-1}; \\ \tau_\sigma &= 0.05\sigma_0 = 0.50 \text{ MPa.} \end{aligned}$$

The curvature function method converged to within τ_σ after 15 iterations. Only one finite element analysis was required in each iteration. Figure 4 shows the iteration histories of the performance index $\Delta\sigma_{\text{abs}}$ and area of the beam. Figure 5 shows the stress distribution of the initial and final shape. Note that in the final design, the stress along the boundary is constant at 10 MPa, except two under stressed zones at both ends of the boundary curve. Figure 6 shows the finite element model for the final design. Figure 7 compares the final shape with eqn (12). The curve generated by the curvature function method appears to be very close to that from the ideal beam theory.

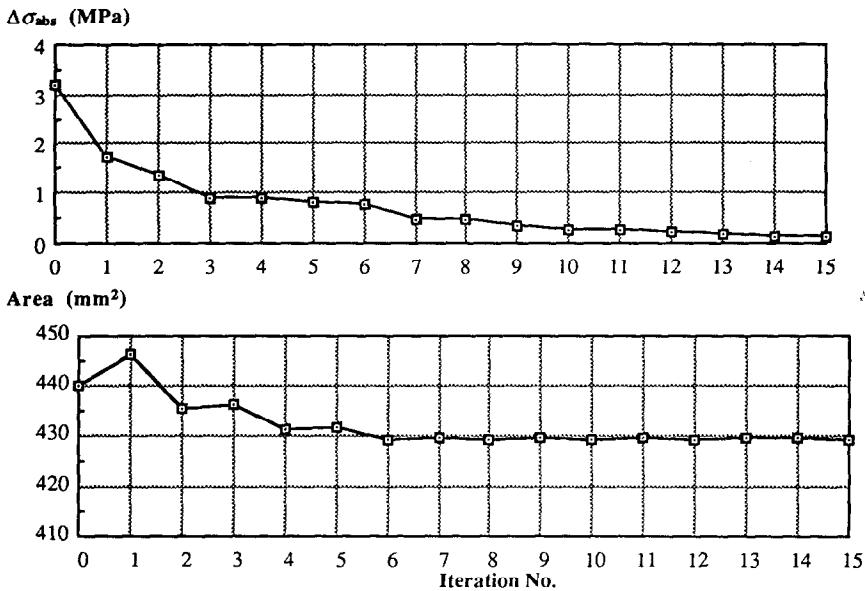


Fig. 4. Beam example: iteration history.

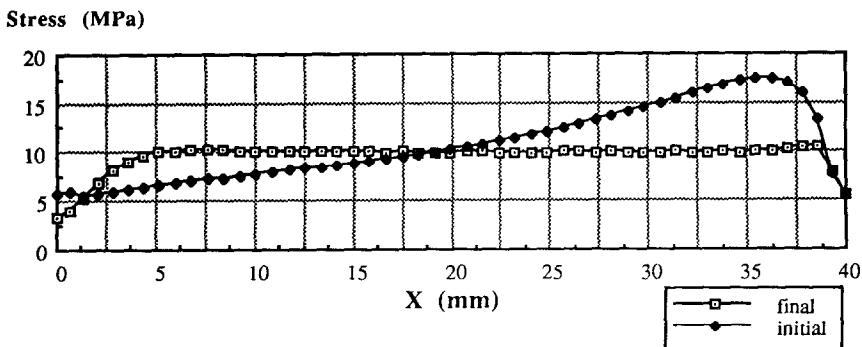


Fig. 5. Beam example: stress distribution of initial and final shape.

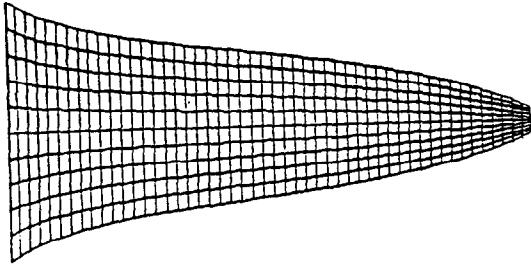


Fig. 6. Beam example: the finite element model for the final shape.

2. Constant stress fillets of a tension bar

A fillet that tapers a tension bar from one section size to another is shown in Fig. 8. Only the upper half of the bar is considered because of symmetry about the $y = -4.5$ in axis. Only the boundary Γ between points A and B is to be optimized. This problem was first proposed by Yang *et al.* [20]. It was also studied in the articles by Shyy *et al.* [21], Rajan and Belegundu [22] and Yang [23]. The task was to find the minimum area fillet with a maximum stress concentration factor $K_t = 1.20$.

The curvature function method was used to find the profiles of the constant stress fillets for $K_t = 1.20$,

1.10, 1.05. The input parameters to the algorithm were:

$$\begin{aligned} \Gamma^0 &= \text{a straight line between point } A \text{ and } B; \\ \sigma_0 &= 10,500 \text{ psi for } K_t = 1.05, 11,000 \text{ psi for } K_t = 1.10, 12,000 \text{ psi for } K_t = 1.20; \\ \alpha^0 &= \frac{1}{6.3} \text{ in}^{-1} \text{ (from eqn (11));} \\ \kappa_{\max} &= 3\alpha^0 = \frac{3}{6.3} \text{ in}^{-1}; \\ \tau_\sigma &= 0.05\sigma_0. \end{aligned}$$

The curvature function method converged to within τ_σ after eight iterations for all three designs. Figure 9 shows the iteration histories of the performance index $\Delta\sigma_{\text{abs}}$ and areas enclosed by the fillet and the X, Y axes. Figure 10 shows the stress distributions of the three final designs. There are acceptable understressed zones at the beginning of the curves. The areas for the whole bar are 132.693 in^2 ($K_t = 1.20$), 133.829 in^2 ($K_t = 1.10$) and 134.410 in^2 ($K_t = 1.05$). Figure 11 shows the finite element model of the final design for $K_t = 1.05$.

Yang *et al.* [20] applied a linearization method on the same problem. Using different finite elements and boundary representations, it took 15–30 iterations to converge to the final design for $K_t = 1.20$. The resulting areas ranged from 133.088 to 135.091 in^2 . Shyy *et al.* [21] used different order p -version finite elements

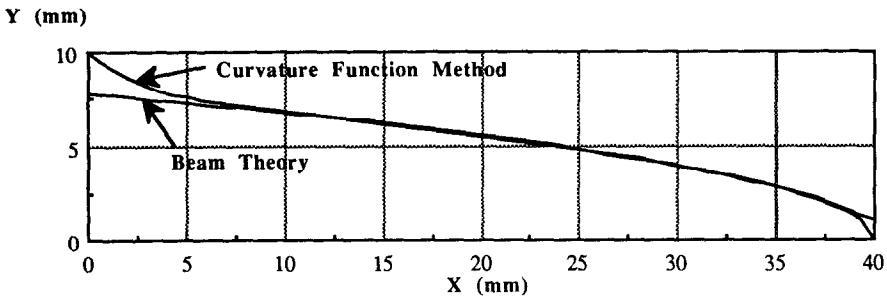
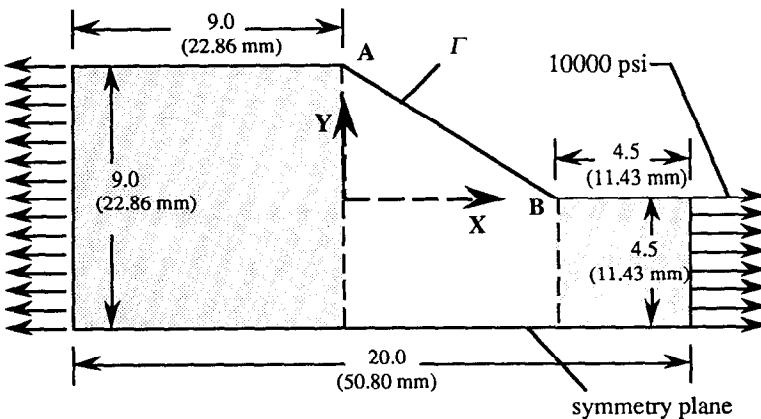


Fig. 7. Comparison of the final curve with that from beam theory.



units: inches unless noted

Fig. 8. Geometric configuration of a fillet.

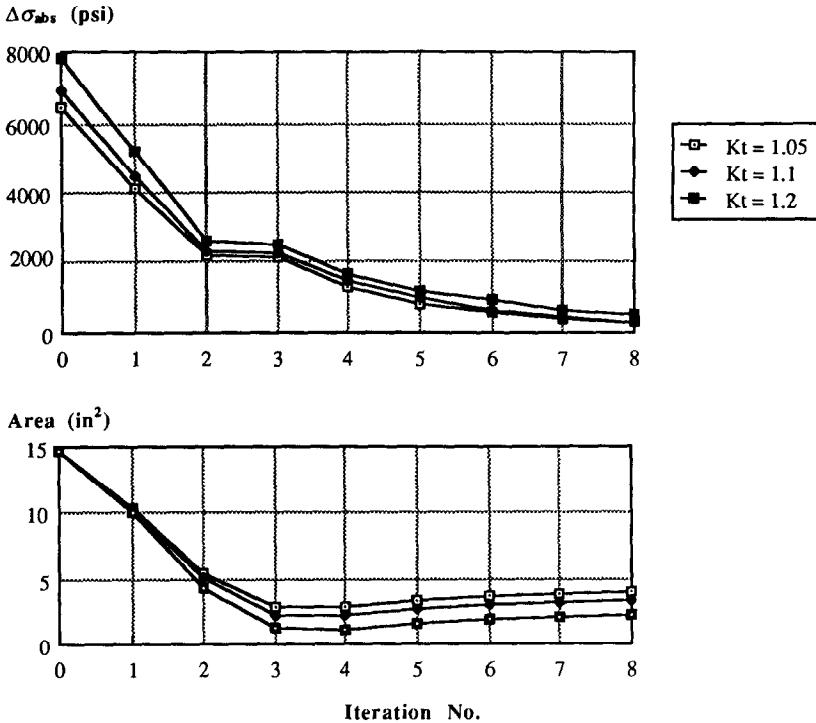


Fig. 9. Fillet example: iteration history.

and CONLIN optimizer on this fillet problem. They reported convergence to $K_t = 1.22$ after four iterations, but the CPU time required for full optimization is more than 11 times that of one finite element analysis. The area of their final design was not reported. Rajan and Belegundu [22] used fictitious load and reported convergence after nine iterations. The final area was 139.80 in². Yang [23] used the boundary element method and CONMIN optimization program on the same problem. He reported convergence after 10 iterations to a final area of 134.29 in².

The number of iterations is not an absolute index for comparing the performance of optimization algorithms because the computational loads for each iteration and termination criteria are different for

each algorithm. However, an iteration for the curvature function method did require the smallest computation effort among all algorithms. Only one finite element analysis was needed for each iteration and no sensitivity analysis was required.

3. Optimum shape design of a torque arm—stress constraint

The initial dimensions and loading conditions of a torque arm are shown in Fig. 12. The arm is constrained around the circumference of the hole located at $x = 41.6$ cm. Only the boundary Γ between point A and B ($0 \leq x \leq 41.6$ cm) was to be varied to minimize the area of the arm. The shaded regions ($x < 0, x > 41.6$) were held to their original dimensions throughout.

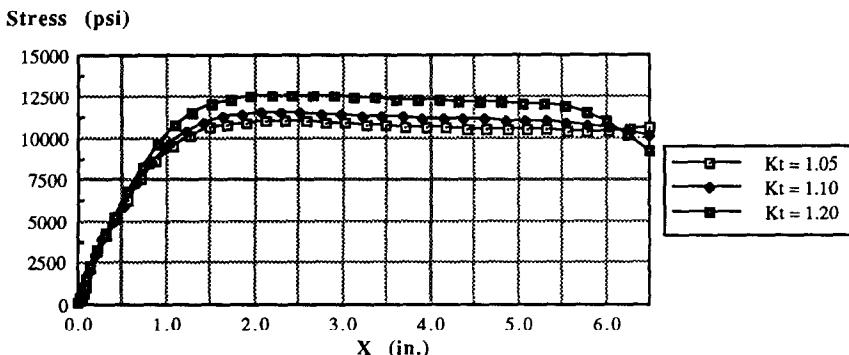


Fig. 10. Fillet example: stress distribution.

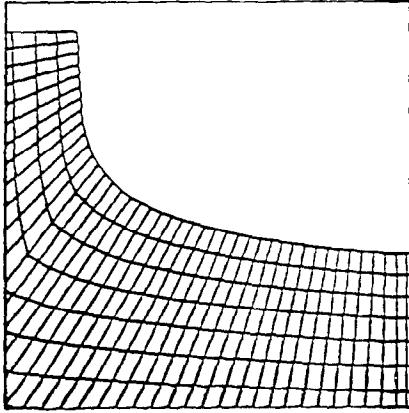


Fig. 11. Fillet example: finite element model of the final design for $K_f = 1.05$.

The original problem, which was proposed by Bokin [5], had a stress and a vertical displacement constraint. This problem was also studied in the articles by Bennett and Bokin [6], Braibant and Fleury [24], Yang *et al.* [20], and Rajan and Belgundu [22]. In this paper, the curvature function

method was used to solve the problem with only the stress constraint. The input parameters to the algorithm are:

$$\begin{aligned} \Gamma^0 &= \text{a straight line between points } A \text{ and } B; \\ \sigma_0 &= 81 \text{ MPa}; \\ \alpha^0 &= \frac{1}{41.6} \text{ cm}^{-1} \text{ [by eqn (11)];} \\ \kappa_{\max} &= 3\alpha = \frac{3}{41.6} \text{ cm}^{-1}; \\ \tau_\sigma &= 0.05\sigma_0 = 4.05 \text{ MPa}. \end{aligned}$$

The curvature function method converged to within τ_σ after 12 iterations. Only one finite element analysis was required in each iteration. Figure 13 shows the iteration histories of the performance index $\Delta\sigma_{abs}$ and area of the arm between points A and B . Figure 14 shows the finite element models during the iterations; the areas are 354.81 (initial shape), 239.64 (fourth iteration), 208.39 (eighth iteration) and 200.19 cm^2 (final shape). The mass of the final design is 0.6093 kg (mass density of steel is $0.0079 \text{ kg cm}^{-3}$). Figure 15 shows the final design and stress distribution along Γ . Note that in Fig. 13 the algorithm oscillated before the sixth iteration. This is because the initial step length α^0 from eqn (11) was too large

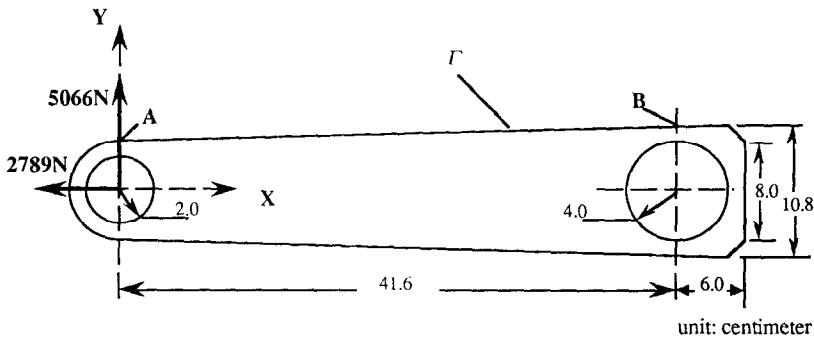


Fig. 12. The dimensions and loading condition of a torque arm.

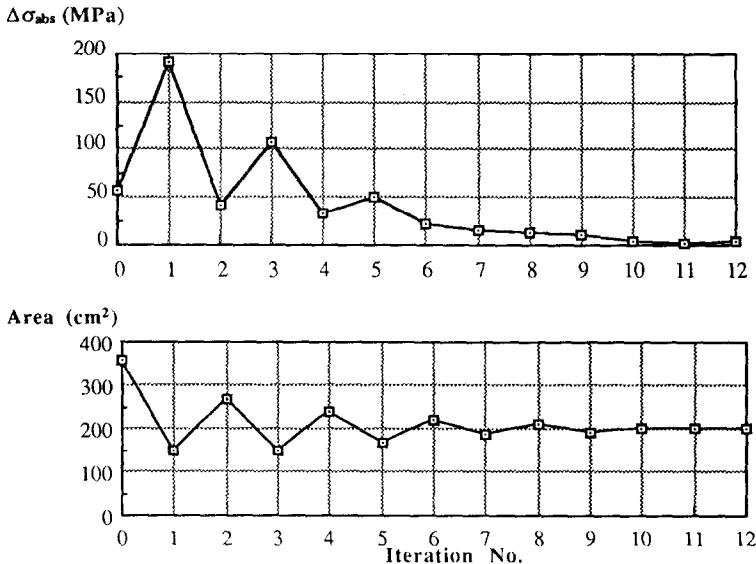
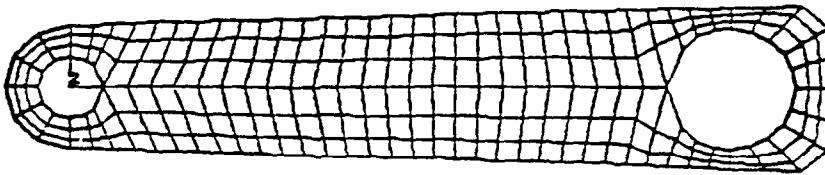
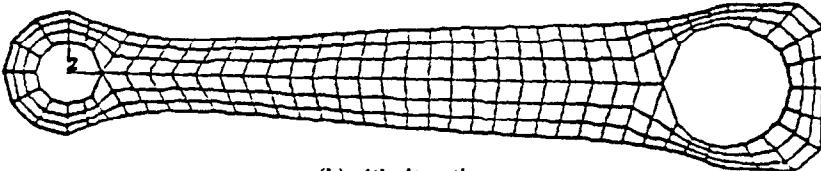


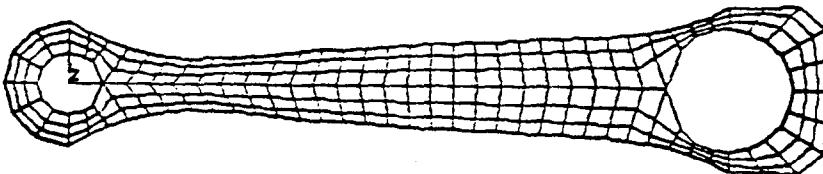
Fig. 13. Torque arm example; iteration history.



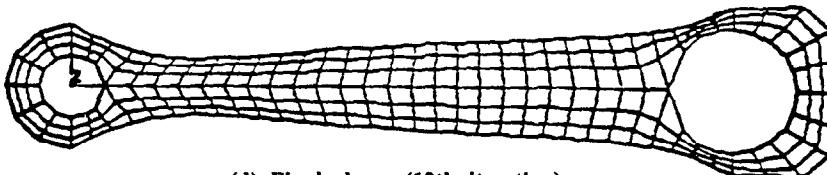
(a) Initial Shape



(b) 4th iteration



(c) 8th iteration



(d) Final shape (12th iteration)

Fig. 14. Torque arm example: finite element models during iterations.

and there are highly understressed zones at both ends of Γ , as shown in Fig. 15. After the sixth iteration, the step length is reduced by the adaptive steplength procedure, and the curvature at the highly understressed zones reaches κ_{\max} , so the algorithm settles down.

The final design generated by the curvature function method cannot be compared with those in the literature because the vertical displacement constraint was relaxed here. The maximum vertical displacement in the final design is 1.23 cm, which violates the constraint (maximum displacement is less than 0.9 cm) in the original problem posed by Bokin [5].

DISCUSSION AND CONCLUSION

The curvature function method for two-dimensional shape optimization under stress constraints is presented in this paper. This method uses the curvatures along the boundary as design variables. The basic idea is to use the strictly monotonic relation between local curvature and stress, instead of purely numerical gradient information, to define the search direction in each iteration. The curvature function method requires only the values of stress, rather than derivatives. It is equally applicable to various analysis techniques, be they finite element methods, boundary element methods, or even physical experiments using

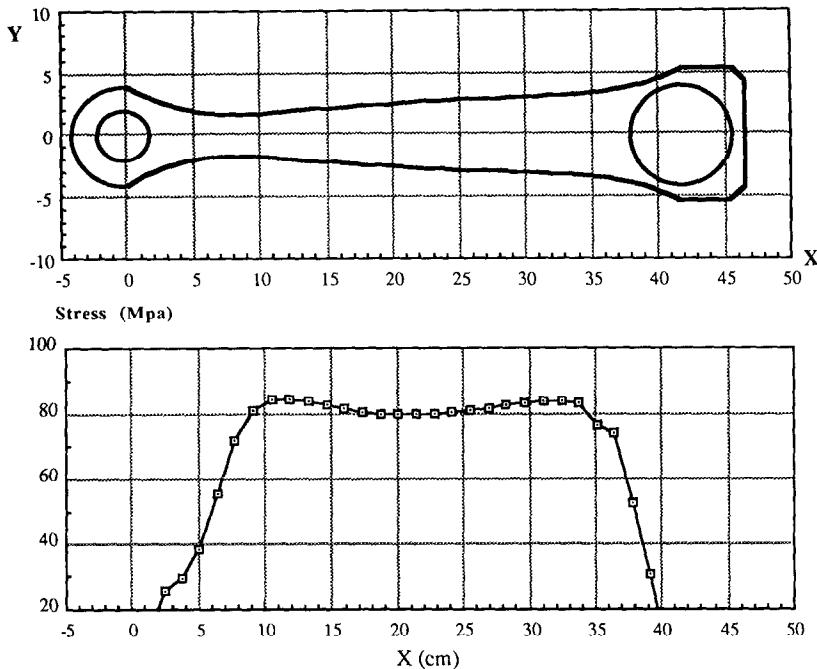


Fig. 15. Torque arm example: stress distribution of the final design.

photo-elasticity or strain gages. Implementation of this method is simple and completely external to the analysis program, no modification of the analysis program is needed.

The design examples illustrate the computational efficiency of the curvature function method. Structural analyses usually dominate the computational load of the shape optimization process. In the curvature function method only one structural analysis is needed in each iteration. Furthermore, this computational load is independent of the number of variables. Thus more variables can be used to describe the shape. In the design examples, up to 50 variables were used to describe the shapes.

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